AN OPTIMAL CONSUMPTION POLICY MODEL FOR ECONOMIC GROWTH

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Abstract. The paper reconsiders the Solow-Swan neoclassical growth model, where, beside capital stocks and labor, the impact of a third production factor is studied. The exhaustible natural resources having its own dynamics will lead to a greater influence of the substitution in production. The problem which arrises is if the conventional evaluation of the national income can be modified as to consider the depletion of the natural resources and the depreciation of the quality of enviroment.

1. INTRODUCTION

In the resources theory there are two intertemporal allocation rules: Hotteling rule and Hartwick rule. The first appears in Dasgupta-Heal-Solow model as a local efficiency condition and stipulates that along a path of efficient employment of resources, the price of exhaustible resources grows by a rate equal to the interest rate. It follows that a null value of the agregate net investments leads to the constancy in time of the consumption. The Hartwick rule was stated for a production economy where consumption at any moment t depends not only on the natural capital extraction, but also on the stock of manufactured capital available at moment t.

Solow and Hartwick have established a sustainability condition of a maximal constant consumption flow in a closed economy which disposes of an exhaustible resource as input, a reproductible capital and uses a constant technology in order to produce consumption goods.

2. THE APPLICATION OF HARTWICK RULE IN A MULTISECTORIAL ECONOMY

Let us consider a two sectors economy. One of them uses as input an exhaustible resource denoted by $s(t)$, $s(0) = s_0 > 0$. The row material $m(t)$ and the capital stock $k(t)$, $k(0) = k_0(t) > 0$ are used as inputs in the other sector. The production function is $y(t) = f(k(t), m(t))$ and the output $y(t)$ is divided between consumption $c(t)$ and accumulation or investments $i(t) = k'(t)$. Let us suppose that there is no depreciation of the capital, all technological progress is endogenous and the level of population is maintained constant. The competitive intertemporal equilibrium provides market prices for the capital goods.

The properties of the consumption function are:

(i) function f is continuous

(ii) function f has positive partial derivatives of first order

$$\frac{\partial f(k,m)}{\partial k} > 0 \text{ and } \frac{\partial f(k,m)}{\partial m} > 0,\ \forall\ k, m > 0$$

(iii) function f has negative partial derivatives of second order

$$\frac{\partial f^2(k,m)}{\partial k^2} < 0 \text{ şi } \frac{\partial f^2(k,m)}{\partial m^2} < 0,\ \forall\ k, m > 0$$

(iv) $\lim_{k\to\infty} f_k(k, m) = 0, m > 0$ and $\lim_{k\to\infty} f_m(k, m) = 0, m > 0$.

Definition 2.1. *A triplet of functions of the form (c(t), k(t), i(t)), where c(t) ≥ 0, m(t) ≥ 0, k(t) ≥ 0, s(t) ≥ 0 and f(k(t), m(t)) ≥ c(t) + i(t), t ≥ 0 is called admisible path for the economic growth.*

Let $\Omega = \{(c(t), k(t), i(t)) |\ t \geq 0\}$ denote the set of all admisible paths. We will presume that following properties hold:

(i) Ω is convex and closed

(ii) if $(c(t), k(t), i(t)) \in \Omega$ then $k \geq 0$

(iii) if $j \leq i$ and $(c, k, i) \in \Omega$ then $(c, k, j) \in \Omega$.

The vector of consumption goods generates utility, described by relation $u(c) = u(c(t))$, where the properties of function u which indicates the utility provided by a level of consumption per capita c are:

(i) function u is differentiable and $\lim_{c\to\infty} u'(c) = 0$

(ii) function u is no decreasing

(iii) function u is concave.

Let $p(t)$ and $q(t)$ denote respectively the present value prices of the consumption goods and capital stocks at moment t. Let $\lambda(t)$ denote the utility discount factor at moment t. If we consider $\lambda(t) = \lambda(0)e^{-\delta t}$ then the discount rate is constant, given by

$$\delta = -\frac{\lambda'(t)}{\lambda(t)} = \frac{\lambda(t)}{\int_t^\infty \lambda(s)ds}.$$

Definition 2.2. *An admisible growth path (c(t), k(t), i(t)), t ≥ 0 is called feasible if k'(t) = i(t),* $k(0) = k_0 > 0$, $s'(t) = -m(t)$, $s(0) = s_0 > 0$.

Definition 2.3. *The feasible path (c(t), k(t), i(t)), t ≥ 0 is said to be competitive relative to the consumption positive discount factors {λ(t)}, t ≥ 0 and to the nonnegative discount prices {p(t), q(t)}, t ≥ 0 if for all t following relations hold:*

(i) *the instant utility λ(t)u(c) – p(t)c is maximized by* $c^*(t)$

(ii) *the instant income is maximized, namely* $(c^*(t), k^*(t), i^*(t))$ *maximizes the expresion p(t)c + q(t)i + q'(t)k for all triplets (c, k, i)* $\in \Omega$.

Remark. The competitive conditions (i) and (ii) can also be expressed as

$$\lambda(t)c^*(t) + p(t)i^*(t) - q(t)m^*(t) + p'(t)k^*(t) + q'(t)s^*(t) \geq$$
$$\geq \lambda(t)c + p(t)i - q(t)m + p'(t)k + q'(t)s$$

Remark. To sustain the fact that $p(t)c + q(t)i + q'(t)k$ denotes the instant profit we will express the prices of the consumption goods and of the production means in current utility terms $P(t) = \frac{p(t)}{\lambda(t)}$ respectively $Q(t) = \frac{q(t)}{\lambda(t)}$. We obtain a nonarbitrarily condition

$$Q'(t) = \frac{d\frac{q(t)}{\lambda(t)}}{dt} = \frac{q'(t)}{\lambda(t)} - q(t)\frac{\lambda'(t)}{\lambda^2(t)} = \frac{q'(t)}{\lambda(t)} + \delta_0(t)Q(t),$$

which proves the applicability of Hotteling rule: along a path of efficient employment of resources, the price of exhaustible resources grows by a rate equal to the interest rate. Further it follows that

$$\frac{p(t)}{\lambda(t)}c + \frac{q(t)}{\lambda(t)}k' + \frac{q'(t)}{\lambda(t)}k = P(t)c + Q(t)k' - \left[\delta_0(t)Q(t) - Q'(t)\right]k,$$

where *P(t)c + Q(t)k'* denotes the current value of production and $[\delta_0(t)Q(t) - Q'(t)]k$ the current cost of capital. Therefore, any competitive path is efficient due to the finite value of the discounted sum of instant utilities. Meanwhile, the transversality condition for the level of capital $\lim_{t\to\infty} q(t)k^*(t) = 0$ holds.

Definition 2.4. *The competitive path* $(c^*(t), k^*(t), i^*(t))$, $t \geq 0$ *is said to be regular relative to the consumption positive discount factors* $\{\lambda(t)\}$, $t \geq 0$ *and to the nonnegative discount prices* $\{p(t), q(t)\}$, $t \geq 0$ *if for all t following relations hold:*

(i) *there exists* $\int_0^\infty \lambda(t)u(c^*(t))dt < \infty$

(ii) $\lim_{t\to\infty} q(t)k^*(t) = 0.$

Definition 2.5. *The feasible path* $(c(t), k(t), i(t))$, $t \geq 0$ *is said to be of type maximin if following relation*

$$\inf_{t\in[0,\infty)} c(t) \geq \inf_{t\in[0,\infty)} c_*(t)$$

holds for all feasible paths $(c_*(t), k_*(t), i_*(t)) \in \Omega$.

Let us consider that in the studied model there exists a path of type maximin $(c(t), k(t), i(t))$, $t \geq 0$ so that

$$\inf_{t\in[0,\infty)} c(t) = c^* > 0.$$

Proposition 2.6. *If the path* $(c^*(t), k^*(t), i^*(t))$, $t \geq 0$ *is regular relative to the consumption positive discount factors* $\{\lambda(t)\}$, $t \geq 0$ *and to the nonnegative discount prices* $\{p(t), q(t)\}$, $t \geq 0$ *then* $(c^*(t), k^*(t), i^*(t))$, $t \geq 0$ *maximizes*

$$\int_0^\infty \lambda(t)u(c^*(t))dt$$

on the set of all admisible consumption plans $k(t)$, $t \geq 0$ *and* $k(0) = k_0$.

Proof. Let $(c(t), k(t), i(t)) \in \Omega$ so that $k(0) = k_0$. According to condition (i) of Definition 2.3 and further condition (ii) it follows that

$$\int_0^T \lambda(t)\big[u(c(t)) - u(c^*(t))\big]dt \leq \int_0^T p(t)\big[c(t) - c^*(t)\big]dt \leq$$

$$\leq \int_0^T \big\{q(t)\big[k'^*(t) - k'(t)\big] + q'(t)\big[k^*(t) - k(t)\big]\big\}dt =$$

$$= q(T)\big[k^*(T) - k(T)\big] - q(0)\big[k^*(0) - k(0)\big] \leq q(T)k^*(T).$$

According to Definition 2.4 the conclusion holds. □

Proposition 2.7. *Following relations hold*

(i) *If the path* $(c^*(t), k^*(t), i^*(t))$, $t \geq 0$ *is competitive and* $c^*(t) > 0$, $\forall t \geq 0$ *then the relation*

$$\lambda(t)\frac{\partial u(c(t))}{\partial c}(c^*(t)) = p(t)$$

holds for all consumption goods

(ii) *If the set* Ω *is smooth and the path* $(c^*(t), k^*(t), i^*(t))$ *is competitive then*

$$p(t)c'^*(t) + \frac{d(q(t)k'(t))}{dt}(k^*(t)) = 0.$$

Proof. (i) follows from condition (i) of Definition 2.3, value c^* being the maximum point of expression $\lambda(t)u(c) - p(t)c$.

(ii) As the elements of set Ω are time invariant, condition (ii) of Definition 2.3 implies

$$p(t)c^*(t+\Delta t)+q(t)k'^*(t+\Delta t)+q'(t)k^*(t+\Delta t)\le p(t)c^*(t)+q(t)k'^*(t)+q'(t)k^*(t).$$

We divide by Δt and let $\Delta t \to 0$. Then

$$p(t)c'^*(t)+q(t)k''^*(t)+q'(t)k'^*(t)=0,$$

which ends the proof. □

Therefore, if there is no exogenous technological progress and the path *(c*(t), $k^*(t)$, $i^*(t)$), t ≥ 0* is competitive and satisfies $\lim_{t\to\infty} q(t)k'^*(t)=0$ then

$$q(t)k'^*(t)=\int_t^\infty \lambda(s)u'(c^*(s))ds.$$

The level of net investments at moment *t* measures the discounted value of the future modifications in terms of utility. The transversality condition for the level of investments $\lim_{t\to\infty} q(t)k'^*(t)=0$ used in Proposition 2.7 derives from the optimality and the regularity of the path obtained for $\lambda(t) = \lambda(0)e^{-\delta t}$, namely considering a constant discount rate δ.

Definition 2.8. *We say that an economy satisfies the Hartwick rule if the discounted value of net investments $q(t)k'^*(t)$ is constant and null.*

Proposition 2.9. *If in an economy with constant level of population and endogenous technology the Hartwick rule holds, then the utility is constant.*

Proof. As in Definition 2.3 and relation $q(t)k'^*(t) = 0$, $\forall t \in (t_1, t_2)$, it follows according to statements (i) respectively (ii) of Proposition 2.7

$$\lambda(t)u'(t)=p(t)c'^*(t)=-\frac{d(q(t)k'^*(t))}{dt}=0, \forall t\in(t_1,t_2).$$

Therefore, the utility is kept constant. □

3. THE GENERALIZED SOLOW - HARTWICK RULE APPLIED TO AN OPTIMAL CONSUMPTION POLICY

In what follows some characteristics of an optimal consumption policy are emphasized. We will show that the current value Hamiltonian does not indicate the maximum sustainable constant consumption flow.

Let us consider a social welfare function defined as the present value of the utility flows of the consumers. The goal is to plan a path of resource extraction and consumption in order to maximize the function given by the sum of instant utilities in the form of consumption, taking into account the exhaustibility of resources. The problem is:

Find

$$\max_{c(t)}\int_0^\infty u(c(t))e^{-\delta t}dt$$

such that $s'(t) = -c(t)$.

The Hamiltonian is given by $H(t, c(t), s(t), \xi(t)) = u(c(t)) - \xi(t)c(t)$, where $\xi(t)$ denotes the costate variable view as the shadow price of the resource stock in units of utility. The first order conditions for an optimal path are given by

$$\frac{\partial H}{\partial c}=u'(c(t))-\xi(t)=0,\ -\frac{\partial H}{\partial s}=\xi'(t)-\delta\xi(t)$$

and the transversality condition $\lim_{t\to\infty}\xi(t)s(t)e^{-\delta t}=0$. The optimal consumption path is given by

$$c'(t) = -\frac{\delta}{\nu(c)} c(t),$$

where $\nu(c) = -\frac{cu''(c)}{u'(c)}$ denotes the elasticity of the marginal utility of consumption.

It follows that an economy with exhaustible resource can not sustain undefinitely a constant flow for the consumption and utility.

If we consider an utility function with constant elasticity, given by $u(c) = \frac{c^{1-\nu}}{1-\nu}$, $0 < \nu < \infty$, along the optimal path consumption decreases in time with a constant rate of $-\frac{\delta}{\nu}$, as in the relation $c(t) = \frac{\delta}{\nu} s_0 e^{-\frac{\delta}{\nu}t}$. If consumption at any moment is given as a fraction of the available resource stock we obtain as level of consumption $c(t) = \frac{\delta}{\nu} s(t)$. It follows that the optimal policy is to extract and consume the fraction $\frac{\delta}{\nu}$ of the remaining stock, $\forall t > 0$.

Proposition 3.1. *The time invariance of the optimal Hamiltonian is a necessary and sufficient condition to sustain undefinitely a constant flow of consumption.*

Proof. Let $H^*(t)$ denote the optimal Hamiltonian at moment t, given by the relation

$$H^*(t) = \delta \int_t^\infty u(c^*(\tau)) e^{-\delta(\tau - t)} d\tau$$

where c^* is the optimal level of consumption. Further it follows that $H^*(t)' = \delta\left[H^*(t) - u(c(t))\right]$. So $H^*(t)' = 0$ implies a constant path of utility $u(c(t)) = H^*(t) = H^*$, $\forall t > 0$.

Reciprocally, it is obvious that if we consider a constant path of utility $u(c^*(\tau)) = u(c^*)$, $\forall \tau \in (t, \infty)$ then the optimal value of the Hamiltonian is stationary $H^*(t) = u(c^*)$, $\forall t > 0$. □

More generally, let us consider the optimal control problem:

Find

$$\max \int_0^\infty f(t, x, u) e^{-\delta t} dt$$

so that

$$x'(t) = g(t, x(t), u(t)),\ x(0) = x_0$$

Proposition 3.2. *The necessary and sufficient condition for a constant consumption and utility flow is a stationary value of the optimal Hamiltonian.*

Proof. Let $x^*(t)$, $u^*(t)$, $\xi^*(t)$ be the solution of the previous problem. The Hamiltonian $H(t, x, u) = f(t, x, u) + \xi g(t, x, u)$ is maximized along these optimal paths. Generally, either for non autonomous or autonomous problems it follows that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial u} u' + \frac{\partial H}{\partial x} x' + \frac{\partial H}{\partial \xi} \xi'.$$

Along the optimal paths following relations hold $\frac{\partial H}{\partial u} u' = 0$, $-\frac{\partial H}{\partial x} = \xi' - \delta\xi$, $x' = \frac{\partial H}{\partial \xi}$. Then

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \delta \xi x'.$$

In the case of an autonomous problem $\frac{\partial H}{\partial t} = 0$ and further $\frac{dH}{dt} = \delta\xi x'$. It follows that $\frac{dH}{dt} = 0$ if and only if $\delta\xi x' = 0$.□

Remark. For a utility function with constant utility we obtain

$$H^*(t) = \frac{\nu}{1-\nu}\left(\frac{\delta}{\nu} s_0\right)^{1-\nu} e^{-\delta\left(\frac{1}{\nu}-1\right)t}$$

and

$$\frac{dH^*(t)}{dt} = -\delta\xi(t)c(t) = -\delta\left(\frac{\delta}{\nu}s_0\right)^{1-\nu} e^{-\delta\left(\frac{1}{\nu}-1\right)t} < 0\,,$$

which means that the constant consumption flow is not sustainable on an indefinit period of time.

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